CAPTURING CLASSICAL TRUTH IN DE MORGAN LOGICS

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A conception of truth

It seems natural, starting from the language $\mathcal{L}$ of arithmetic expanded with a unary truth predicate T, to endorse the following conditions:

1. $s = t$ is true iff the values of s and t coincide, false otherwise.
2. A conjunction $\phi \wedge \psi$ is true iff ϕ is true and ψ is true; false iff $\neg\phi$ or $\neg\psi$ are true.
3. $\forall x \phi$ is true iff $\phi(t)$ is true for all closed terms, false if $\neg\phi(s)$ is true for some s.
4. $\neg\phi$ is true iff ϕ is false, and false if ϕ is true.
5. $\mathsf{T}\ulcorner\phi\urcorner$ is true iff ϕ is true, and false iff ϕ is false.

Basic De Morgan Logic

(IN) $\Gamma \Rightarrow \Delta$, if $\Gamma \cap \Delta \neq \varnothing$

(Cut) $$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

(W1) $$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}$$

(W2) $$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}$$

(Sub) $$\frac{\Gamma \Rightarrow \Delta}{\Gamma(t/x) \Rightarrow \Delta(t/x)}$$

($\neg$) $$\frac{\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\Gamma}$$

(= 1) $\Rightarrow t = t$

(= 2) $s = t, \phi(s/x) \Rightarrow \phi(t/x)$

($\wedge$) $$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}$$

($\vee$1) $$\frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}$$

($\forall$) $$\frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \forall x\phi, \Delta}$$

($\exists$) $$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \exists x\phi \Rightarrow \Delta}$$

...and classical logic

Just add:

$$(\neg \mathsf{k1}) \quad \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma, \neg\, \phi \Rightarrow \Delta} \qquad (\neg \mathsf{k2}) \quad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\phi, \Delta}$$

Arithmetic

ARITHMETIC

Sequents $\Rightarrow \phi$ for ϕ a basic axioms of PA and

(IND)
$$\frac{\Gamma, \phi(x) \Rightarrow \phi(Sx), \Delta}{\Gamma, \phi(\overline{0}) \Rightarrow \phi(t), \Delta}$$

with $\phi \in \mathcal{L}_{\mathsf{T}}$ and x eigenvariable.

We need to be careful with extending or not induction to T.

Axiomatization in BDM: PKF

PKF1

1. $s^\circ = t^\circ \Leftrightarrow \mathsf{T}(s \underset{\cdot}{=} t)$

PKF2

1. $\mathrm{Sent}_{\mathcal{L}_T}(x \underset{\cdot}{\wedge} y), \mathsf{T}(x \underset{\cdot}{\wedge} y) \Rightarrow \mathsf{T}x \wedge \mathsf{T}y$
2. $\mathrm{Sent}_{\mathcal{L}_T}(x \underset{\cdot}{\wedge} y), \mathsf{T}x \wedge \mathsf{T}y \Rightarrow \mathsf{T}(x \underset{\cdot}{\wedge} y)$

PKF3

1. $\mathrm{Sent}_{\mathcal{L}_T}(x), \mathsf{T}\underset{\cdot}{\neg}\underset{\cdot}{\neg}x \Rightarrow \mathsf{T}x$
2. $\mathrm{Sent}_{\mathcal{L}_T}(x), \mathsf{T}x \Rightarrow \mathsf{T}\underset{\cdot}{\neg}\underset{\cdot}{\neg}x$

PKF6

1. $\mathsf{T}t^\circ \Leftrightarrow \mathsf{T}\underset{\cdot}{\mathsf{T}}t$

PKF7

1. $\mathrm{Sent}_{\mathcal{L}_T}(x), \neg\mathsf{T}x \Rightarrow \mathsf{T}\underset{\cdot}{\neg}x$
2. $\mathrm{Sent}_{\mathcal{L}_T}(x), \mathsf{T}\underset{\cdot}{\neg}x \Rightarrow \neg\mathsf{T}x$

PKF8

$\mathsf{T}x \Rightarrow \mathrm{Sent}_{\mathcal{L}_T}(x)$

Axiomatization in classical logic: KF

KF1 (1) $s^\circ = t^\circ \Leftrightarrow \mathsf{T}(s \dot{=} t)$

KF2 (1) $s^\circ \neq t^\circ \Leftrightarrow \mathsf{T}(\dot{\neg} s = t)$

KF3 (1) $\mathrm{Sent}_{\mathcal{L}_T}(x), \mathsf{T}\dot{\neg}\dot{\neg} x \Rightarrow \mathsf{T}x$
(2) $\mathrm{Sent}_{\mathcal{L}_T}(x), \mathsf{T}x \Rightarrow T\dot{\neg}\dot{\neg} x$

KF4 (1) $\mathrm{Sent}_{\mathcal{L}_T}(x \dot{\wedge} y), \mathsf{T}(x \dot{\wedge} y) \Rightarrow \mathsf{T}x \wedge \mathsf{T}y$
(2) $\mathrm{Sent}_{\mathcal{L}_T}(x \dot{\wedge} y), \mathsf{T}x \wedge \mathsf{T}y \Rightarrow \mathsf{T}(x \dot{\wedge} y)$

KF5 (1) $\mathrm{Sent}_{\mathcal{L}_T}(x \dot{\wedge} y), \mathsf{T}\dot{\neg}(x \dot{\wedge} y) \Rightarrow \mathsf{T}\dot{\neg} x \vee \mathsf{T}\dot{\neg} y$
(2) $\mathrm{Sent}_{\mathcal{L}_T}(x \dot{\wedge} y), \mathsf{T}\dot{\neg} x \vee \mathsf{T}\dot{\neg} y \Rightarrow \mathsf{T}\dot{\neg}(x \dot{\wedge} y)$

KF12 $\mathsf{T}t^\circ \Leftrightarrow \mathsf{T}\dot{\mathsf{T}}t$

KF13 $\mathsf{T}\dot{\neg}\dot{\mathsf{T}}t \Leftrightarrow \mathsf{T}\dot{\neg} t^\circ \vee \neg \mathrm{Sent}_{\mathcal{L}_T}(t^\circ)$

KF14 $\mathsf{T}x \Rightarrow \mathrm{Sent}_{\mathcal{L}_T}(x)$

Comparing the classical and the nonclassical

COMPARE WHAT THE THEORIES PROVE TRUE

Internal 'theory' of S:

$$\mathsf{I}S := \{\phi \in \mathcal{L}_{\mathsf{T}} \mid S \vdash \mathsf{T}\ulcorner\phi\urcorner\}$$

When induction is extended

PROPOSITION (HALBACH&HORSTEN 2006)

PKF proves true the same arithmetical sentences as $\mathsf{RT}_{<\omega^\omega}$.

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The internal logics of the theories differ considerably.

When only truth is at stake

PROPOSITION (HALBACH&NICOLAI 2016)

If $\mathsf{KF}\restriction \vdash T\ulcorner\phi\urcorner$, then $\mathsf{PKF}\restriction \vdash \phi$.

COROLLARY

$\mathsf{IKF}\restriction = \mathsf{PKF}\restriction$

The Project

There is a clear indication that the interaction between the logic and the induction principle is responsible for the asymmetry. We now study how to vary the induction rules to find the right counterparts to the internal theories of PKF and KF.

Ordinals

We assume a standard ordinal notation up to ε_0 in PKF:

- $\alpha \xrightarrow{1\text{-}1} \ulcorner \alpha \urcorner$, with $\alpha < \varepsilon_0$;
- a p.r. relation $\ulcorner \alpha \urcorner \prec \ulcorner \beta \urcorner :\Leftrightarrow \alpha < \beta$;
- p.r. functions

$$\ulcorner \alpha \urcorner \oplus \ulcorner \beta \urcorner := \ulcorner \alpha + \beta \urcorner$$
$$\ulcorner \alpha \urcorner \otimes \ulcorner \beta \urcorner := \ulcorner \alpha \times \beta \urcorner$$
$$\widetilde{\omega}^{\ulcorner \alpha \urcorner} := \ulcorner \omega^{\alpha} \urcorner$$

Adding Mathematical Patterns of Reasoning to PKF

We add to PKF a rule of transfinite induction up to any $\alpha < \varepsilon_0$.

For any $\alpha < \varepsilon_0$ and $\phi \in \mathcal{L}_{\mathsf{T}}$,

$\mathsf{TI}_{\mathcal{L}_{\mathsf{T}}}(< \varepsilon_0)$

$$\frac{\Gamma, \forall \xi \prec \beta\, \phi(\xi) \Rightarrow \phi(\beta), \Delta}{\Gamma \Rightarrow \forall \xi \prec \alpha\, \phi(\xi), \Delta}$$

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$$\frac{\Gamma, \forall \xi \prec \beta\ \phi(\xi) \Rightarrow \phi(\beta), \Delta}{\Gamma \Rightarrow \forall \xi \prec \alpha\ \phi(\xi), \Delta}$$

We call the resulting system PKF^+.

Hierarchies

How many 'classical' truth predicates can be captured in PKF and PKF^+?

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THE LANGUAGES $\mathcal{L}_\alpha$

We let

$$\mathcal{L}_0 := \mathcal{L}$$

$$\mathcal{L}_{\alpha+1} := \mathcal{L}_\alpha \cup \{\mathsf{T}_\alpha\}, \text{ with } \mathsf{T}_\alpha(\ulcorner\phi\urcorner) :\leftrightarrow \mathsf{T}\ulcorner\phi\urcorner \wedge Sent_{\mathcal{L}_\alpha}(\ulcorner\phi\urcorner)$$

$$\mathcal{L}_{\varepsilon_0} := \bigcup_{\alpha<\varepsilon_0} \mathcal{L}_\alpha$$

Hierarchies

How many 'classical' truth predicates can be captured in PKF and PKF$^+$?

LEMMA (ESSENTIALLY HALBACH&HORSTEN 2006)

1. $\mathsf{PKF} \vdash (\forall x : Sent_{\mathcal{L}_\alpha})(\mathsf{T}x \vee \neg \mathsf{T}x)$ for all $\alpha < \omega^\omega$;
2. $\mathsf{PKF}^+ \vdash (\forall x : Sent_{\mathcal{L}_\alpha})(\mathsf{T}x \vee \neg \mathsf{T}x)$ for all $\alpha < \varepsilon_0$.

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Proof Idea.

Induction on α and additional side induction on the complexity of the sentence involved.

Hierarchies

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2. $\mathsf{PKF}^+ \vdash (\forall x : Sent_{\mathcal{L}_\alpha})(\mathsf{T}x \vee \neg \mathsf{T}x)$ for all $\alpha < \varepsilon_0$.

Proof Idea.

In particular, in PKF and PKF^+,

$$\forall \xi \prec \alpha\, (\mathsf{T}_\xi x \vee \neg \mathsf{T}_\xi x) \;\Rightarrow\; \mathsf{T}_\alpha x \vee \neg \mathsf{T}_\alpha x$$

then one applies $\mathsf{TI}_{\mathcal{L}_\mathsf{T}}(< \omega^\omega)$ – provable in PKF^+ – or $\mathsf{TI}_{\mathcal{L}_\mathsf{T}}(< \varepsilon_0)$.

Internal induction: the theory $\mathsf{KF_{int}}$

DEFINITION ($\mathsf{KF_{int}}$)

The basic axioms of KF with the rule

$$\frac{\Gamma, \mathsf{T}\ulcorner\phi(\dot{x})\urcorner \Rightarrow \mathsf{T}\ulcorner\phi(S\dot{x})\urcorner, \Delta}{\Gamma, \mathsf{T}\ulcorner\phi(0)\urcorner \Rightarrow \mathsf{T}\ulcorner\phi(\dot{y})\urcorner, \Delta}$$

with the usual restrictions on y.

$\mathsf{PKF}^+ \subseteq \mathsf{IKF}$ *and* $\mathsf{PKF} \subseteq \mathsf{KF}_{\mathsf{int}}$

LEMMA

- If $\mathsf{PKF} \vdash \mathsf{T}\ulcorner\phi\urcorner$, then $\mathsf{KF}_{\mathsf{int}} \vdash \mathsf{T}\ulcorner\phi\urcorner$;
- if $\mathsf{PKF}^+ \vdash \mathsf{T}\ulcorner\phi\urcorner$, then $\mathsf{KF} \vdash \mathsf{T}\ulcorner\phi\urcorner$.

Proof Idea.

One proves that if $\mathsf{PKF} \vdash \Gamma \Rightarrow \Delta$ (and similarly for PKF^+),

1. $\mathsf{KF}_{\mathsf{int}} \vdash \forall\vec{t}\left(T\ulcorner\bigwedge\Gamma\urcorner(\vec{t}/\vec{v}) \rightarrow T\ulcorner\bigvee\Delta\urcorner(\vec{t}/\vec{v})\right)$ (resp. KF)
2. $\mathsf{KF}_{\mathsf{int}} \vdash \forall\vec{t}\left(T\ulcorner\neg\bigvee\Delta\urcorner(\vec{t}/\vec{v}) \rightarrow T\ulcorner\neg\bigwedge\Gamma\urcorner(\vec{t}/\vec{v})\right)$ (resp. KF)

where $\bigwedge\varnothing :\leftrightarrow 0 = 0$ and $\bigvee\varnothing :\leftrightarrow 0 \neq 0$, and $\bigwedge\Gamma(\bigvee\Gamma)$ is the conjunction (disjunction) of all sentences in Γ.

$\mathsf{PKF}^+ \subseteq \mathsf{IKF}$ *and* $\mathsf{PKF} \subseteq \mathsf{KF}_{\mathsf{int}}$

LEMMA

- If $\mathsf{PKF} \vdash \mathsf{T}\ulcorner\phi\urcorner$, then $\mathsf{KF}_{\mathsf{int}} \vdash \mathsf{T}\ulcorner\phi\urcorner$;
- if $\mathsf{PKF}^+ \vdash \mathsf{T}\ulcorner\phi\urcorner$, then $\mathsf{KF} \vdash \mathsf{T}\ulcorner\phi\urcorner$.

Proof Idea.

For PKF^+ one notices that KF proves $\mathsf{TI}_{\mathcal{L}_\mathsf{T}}(<\varepsilon_0)$.

COROLLARY

$\mathsf{PKF}^+ \subseteq \mathsf{IKF}$ and $\mathsf{PKF} \subseteq \mathsf{IKF}_{\mathsf{int}}$.

CLAIM

$\mathsf{IKF}_{\mathsf{int}} \subseteq \mathsf{PKF}$ and $\mathsf{IKF} \subseteq \mathsf{PKF}^+$.

Kripke-Feferman again

TKF and $\mathsf{TKF}_{\mathsf{int}}$ are the versions of KF and $\mathsf{KF}_{\mathsf{int}}$ in a one-sided sequent calculus à la Tait.

Kripke-Feferman again

TKF and $\mathsf{TKF}_{\mathsf{int}}$ are the versions of KF and $\mathsf{KF}_{\mathsf{int}}$ in a one-sided sequent calculus à la Tait.

We define TKF^{∞} as usual, by *omitting number variables*, and adding for $\phi \in \mathcal{L}_{\mathsf{T}}$:

$$(\omega \mathsf{R}) \qquad \frac{\Gamma, \phi(t) \;\text{ for all cterms } t}{\Gamma, \forall x\, \phi(x)}$$

We assume the standard definitions of *surface complexity*, *length* of a derivation, and *cut rank* of a formula. We write as usual $W \vdash^{\mathsf{lh}}_{\mathsf{cr}} \Gamma$ for derivations in W with the mentioned measures.

Partial Cut Elimination

Quasi-normal derivations have cut rank 0: the cut rule is applied only to atomic formulas (incl. $\mathsf{T}t, \neg\mathsf{T}t$).

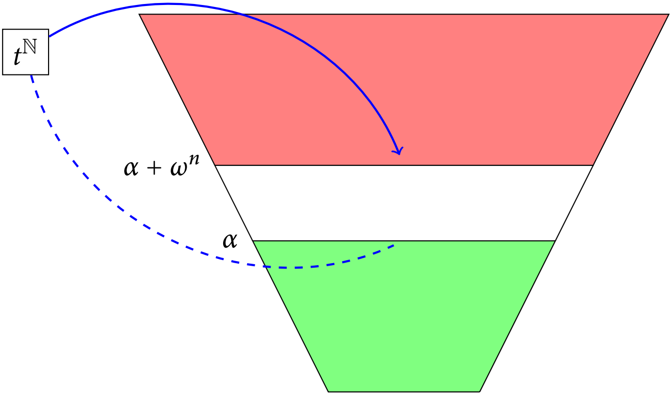
LEMMA

If $\mathsf{TKF}_{\mathsf{int}} \vdash^n_k \Gamma$, then $\mathsf{TKF}_{\mathsf{int}} \vdash^{2^n_k} \Gamma$.

Asymmetric Interpretation: Cantini (1989)

For all $\alpha > 0$:

- If $\mathsf{TKF}_{\mathsf{int}} \vdash^n \mathsf{T}t$, then $t^{\mathbb{N}}$ is definitely true at $\alpha + \omega^n$;
- If $\mathsf{TKF}_{\mathsf{int}} \vdash^n \neg\mathsf{T}t$, then $t^{\mathbb{N}}$ is not yet true at α.



PROPOSITION

If $\mathsf{TKF}_{\mathsf{int}} \vdash \mathsf{T}\ulcorner\phi\urcorner$, then $\mathsf{PKF} \vdash \phi$.

PROPOSITION

If $\mathsf{TKF_{int}} \vdash \mathsf{T}\ulcorner\phi\urcorner$, then $\mathsf{PKF} \vdash \phi$.

Proof.

- We first apply partial cut elimination to obtain a quasi-normal derivation.
- Reasoning in PKF, if $\mathsf{TKF_{int}} \vdash^n \mathsf{T}\ulcorner\phi\urcorner$, then $\mathsf{T}_\alpha\ulcorner\phi\urcorner$ for $\alpha < \omega^\omega$, where

$$\mathsf{T}_\alpha\ulcorner\phi\urcorner :\leftrightarrow \mathsf{T}\ulcorner\phi\urcorner \wedge Sent_{\mathcal{L}_\alpha}(\ulcorner\phi\urcorner)$$

- By symmetry, $\mathsf{PKF} \vdash \phi$.

Embedding

Lemma (Embedding)

If TKF $\vdash \Gamma$, then there is an $n \in \omega$ and an $\alpha < \omega^2$ such that $\mathsf{TKF}^\infty \vdash^\alpha_n \Gamma$.

Embedding

LEMMA (EMBEDDING)

If $\mathsf{TKF} \vdash \Gamma$, then there is an $n \in \omega$ and an $\alpha < \omega^2$ such that $\mathsf{TKF}^\infty \vdash^\alpha_n \Gamma$.

Since instances of induction become 'logically' valid, we have

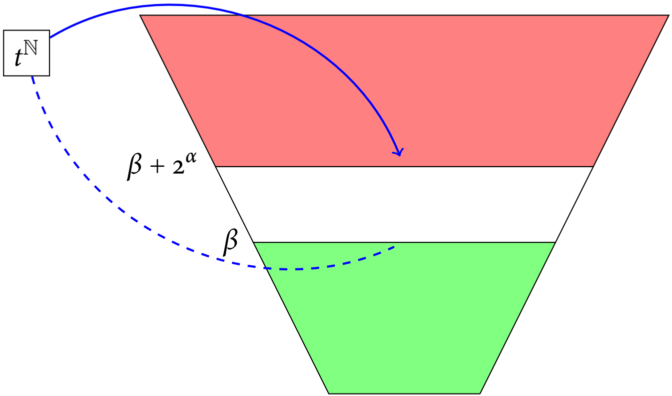
LEMMA (PARTIAL CUT ELIMINATION)

1. If $\mathsf{TKF}^\infty \vdash^\alpha_{n+1} \Gamma$, then $\mathsf{TKF}^\infty \vdash^{2^\alpha}_n \Gamma$;
2. If $\mathsf{TKF}^\infty \vdash^\alpha_{n+1} \Gamma$, then $\mathsf{TKF}^\infty \vdash^\beta \Gamma$ for some $\beta < \varepsilon_0$.

Asymmetric Interpretation: Cantini (1989)

For all $\beta > 0$:

- If $\mathsf{TKF} \vdash_0^\alpha \mathsf{T}t$ with $\alpha < \varepsilon_0$, then $t^{\mathbb{N}}$ is definitely true at $\beta + 2^\alpha$;
- If $\mathsf{TKF} \vdash_0^\alpha \neg\mathsf{T}t$ with $\alpha < \varepsilon_0$, then $t^{\mathbb{N}}$ is not yet true at β.



$\mathsf{IKF} \subseteq \mathsf{PKF}^+$

PROPOSITION

If $\mathsf{TKF} \vdash \mathsf{T}\ulcorner\phi\urcorner$, then $\mathsf{PKF}^+ \vdash \phi$.

Proof.

Assume $\mathsf{TKF} \vdash_k^m T\ulcorner\phi\urcorner$. Now we reason in PKF^+ (recall we have all classical truth predicates for $\alpha < \varepsilon_0$):

- embedding and partial cut elimination, $\mathsf{TKF}^\infty \vdash^\alpha T\ulcorner\phi\urcorner$ for some $\alpha < \varepsilon_0$.
- Therefore $\mathsf{T}_\beta\ulcorner\phi\urcorner$ for some $\beta < \varepsilon_0$.

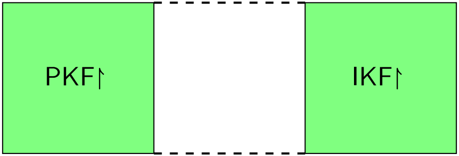
Now back in the real world,

$$\mathsf{PKF}^+ \vdash \mathsf{T}\ulcorner\phi\urcorner$$

□

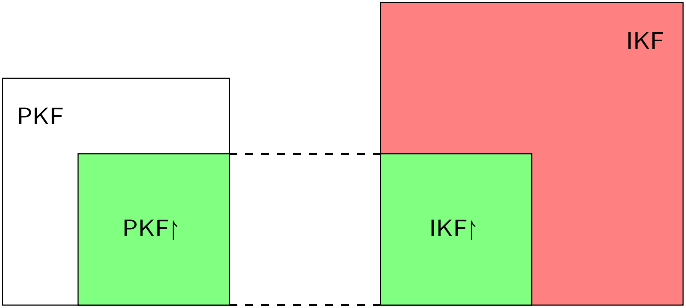
What can we prove true?

No interaction between induction and truth:



What can we prove true?

Induction is open-ended:



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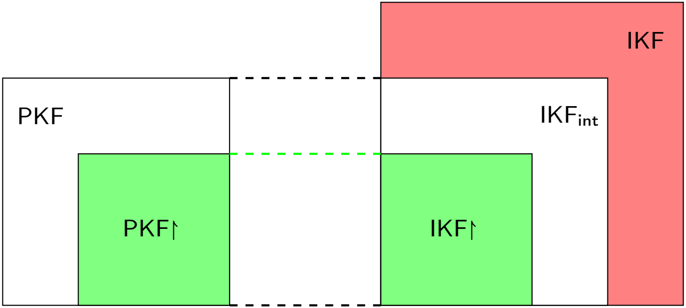

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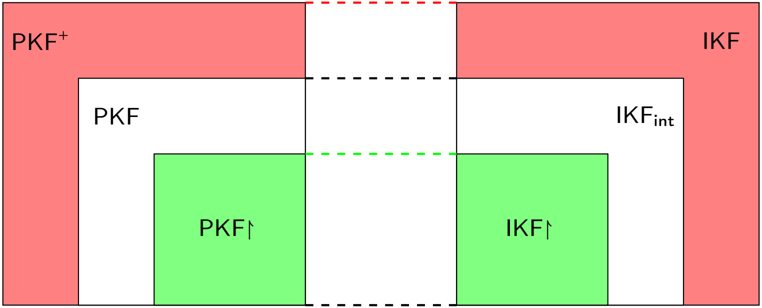

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TKF

- Standard logical axioms and rules of the Tait calculus;
- Defining equations for function symbols in $\mathcal{L}^+$;
- TKF special axioms:

(=1)	$\Gamma, \neg T(s \dot{=} t), s^\circ = t^\circ$
(=2)	$\Gamma, \neg T \dot{\neg}(s \dot{=} t), s^\circ \neq t^\circ$
(CONS)	$\Gamma, \neg Tt, \neg T \dot{\neg} t$
(J)	$\Gamma, \neg Tt, Sent_{\mathcal{L}^+}(t)$

TKF

- Special rules of TKF

$$(\mathrm{T})\ \frac{\Gamma, (\neg)\, Tt^{\circ}}{\Gamma, (\neg)\, T\underset{\cdot}{T}t} \qquad (\mathrm{F})\ \frac{\Gamma, (\neg)\, T\underset{\cdot}{\neg}t^{\circ}}{\Gamma, (\neg)\, T\underset{\cdot}{\neg}\,\underset{\cdot}{T}t} \qquad (\neg)\ \frac{\Gamma, Sent_{\mathcal{L}^+}(x) \qquad \Gamma, (\neg)\, Tx}{\Gamma, (\neg)\, T\underset{\cdot}{\neg}\underset{\cdot}{\neg}x}$$

$$(\wedge 1)\ \frac{\Gamma, Sent_{\mathcal{L}^+}(x \underset{\cdot}{\wedge} y) \qquad \Gamma, (\neg)\, (Tx \wedge Ty)}{\Gamma, (\neg)\, T\, x \underset{\cdot}{\wedge} y}$$

$$(\wedge 2)\ \frac{\Gamma, Sent_{\mathcal{L}^+}(x \underset{\cdot}{\wedge} y) \qquad \Gamma, (\neg)\, (T\underset{\cdot}{\neg}x \vee T\underset{\cdot}{\neg}y)}{\Gamma, (\neg)\, T\underset{\cdot}{\neg}\, x \underset{\cdot}{\wedge} y}$$

$$(\forall 1)\ \frac{\Gamma, Sent_{\mathcal{L}^+}(\underset{\cdot}{\forall} vx) \qquad \Gamma, (\neg)\, \forall t\, T\, x(t/v)}{\Gamma, (\neg)\, T(\underset{\cdot}{\forall} vx)}$$

$$(\forall 2)\ \frac{\Gamma, Sent_{\mathcal{L}^+}(\underset{\cdot}{\forall} vx) \qquad \Gamma, (\neg)\, \exists t\, T\underset{\cdot}{\neg}\, x(t/v)}{\Gamma, (\neg)\, T\underset{\cdot}{\neg}\underset{\cdot}{\forall} vx}$$

$$(\mathrm{IND}^{k})\ \frac{\Gamma, \phi(\overline{0}) \qquad \Gamma, \forall x\, (\phi(x) \to \phi(Sx))}{\Gamma, \forall x\, \phi(x)}$$